Journal of Analysis and Applications
Vol. 21 (2023), No.1, pp.1-12
ISSN: 0972-5954

URL : www.sasip.net

Properties of (p, q)-differential equations with (p, q)-Euler polynomials as solutions

C.H. Yu and J.Y. Kang*

Abstract. In this paper, we discuss (p, q)-differential equations which are related to (p, q)-Euler polynomials. Also, we find a basic symmetric property for (p, q)-differential equation using the generating function of (p, q)-Euler polynomials.

AMS Subject Classification (2020): 05A19, 11B83, 34A30, 65L99

Keywords: (p, q)-Euler polynomials, (p, q)-derivative, (p, q)-differential equation

1. Introduction

In 1991, [3] introduced the (p, q)-number in order to unify varied forms of q-oscillator algebras in physics literature. Wachs and White [13] introduced the (p, q)-numbers in mathematics literature in certain combinatorial problems without any connection to the quantum group related to mathematics and physics literature, see [2], [7], [8], [13].

For any $n \in \mathbb{C}$, the (p, q)-number is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

Thereby, several physical and mathematical problems lead to the necessity of (p, q)-calculus. Based on the aforementioned papers, many mathematicians and physicists have developed the (p, q)-calculus in many different research areas, see [4].

*Corresponding author

Definition 1.1 [1], [12]. Let z be any complex numbers with $|z| < 1$. The two forms of (p,q)-exponential functions are defined by

$$e_{p,q}(z) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{z^n}{[n]_{p,q}!},$$
$$E_{p,q}(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_{p,q}!}.$$

In [5], Corcino made the theorem of (p,q)-extension of binomials coefficients and found various properties which are related to horizontal function, triangular function, and vertical function.

Definition 1.2 [5]. Let $n \geq k$. (p,q)-Gauss Binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!},$$

where $[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q} \cdots [1]_{p,q}$.

Definition 1.3 [1], [12]. (p,q)-derivative operator of any function f, also referred to as the Jackson derivative, is defined the as follows:

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \qquad x \neq 0,$$

and $D_{p,q}f(0) = f'(0)$.

Let $p = 1$ in Definition1.3. Then, we can remark

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \qquad x \neq 0,$$

we call D_q is the q-derivative.

Theorem 1.4 [1], [9]. *The operator,$D_{p,q}$, has the following basic properties:*

i Derivative of a product

$$\begin{aligned} D_{p,q}(f(x)g(x)) &= f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x) \\ &= g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x). \end{aligned}$$

ii Derivative of a ratio

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)}$$
$$= \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)}.$$

In 2016, Araci et al. [1] introduced a new class of Bernoulli, Euler and Genocchi polynomials based on the theory of (p, q)-numbers and found some properties and identities. After that, several studies have investigated the special functions for various applications, see [6], [9], [10], [11].

Definition 1.5 [6]. (p, q)-Euler numbers $\mathcal{E}_{n,p,q}$ and polynomials $\mathcal{E}_{n,p,q}(x)$ are defined by

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}\frac{t^n}{[n]_{p,q}!} = \frac{2}{e_{p,q}(t)+1},$$
$$\sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} = \frac{2}{e_{p,q}(t)+1}e_{p,q}(tx).$$

Consider $p = 1$ in Definition 1.5. Then, we note

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,q}\frac{t^n}{[n]_{q}!} = \frac{2}{e_{q}(t)+1},$$
$$\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x)\frac{t^n}{[n]_{q}!} = \frac{2}{e_{q}(t)+1}e_{q}(tx),$$

where $\mathcal{E}_{n,q}$ is the q-Euler number and $\mathcal{E}_{n,q}(x)$ is the q-Euler polynomial.

2. (p, q)-differential equations which is related to (p, q)-Euler polynomials

Theorem 2.1. *Let n be a non-negative integer. Then, we have*

$$D_{p,q,x}\mathcal{E}_{n,p,q}(x) = [n]_{p,q}\mathcal{E}_{n-1,p,q}(px).$$

Proof. From the generating function of (p,q)-Euler numbers and polynomials, we find

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}&=\sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}p^{\binom{n}{2}}x^n\frac{t^n}{[n]_{p,q}!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}x^{n-k}\mathcal{E}_{k,p,q}\right)\frac{t^n}{[n]_{p,q}!}.\end{aligned}$$

By comparing the coefficients of both sides in the above equation, we have a relation as

$$\mathcal{E}_{n,p,q}(x)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}x^{n-k}\mathcal{E}_{k,p,q}. \tag{1}$$

Using the q-derivative in Equation (1), we obtain as the follows.

$$\begin{aligned}D_{p,q,x}^{(1)}\mathcal{E}_{n,p,q}(x)&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}[n-k]_{p,q}p^{\binom{n-k}{2}}x^{n-k-1}\mathcal{E}_{k,p,q}\\&=[n]_{p,q}\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}_{p,q}p^{\binom{n-k-1}{2}}(px)^{n-k-1}\mathcal{E}_{k,p,q}.\end{aligned}$$

From the above equation, we complete the proof of Theorem 2.1. □

Corollary 2.2. *Let k be a non-negative integer and $0<p/q<1$. Then, the following holds*

$$D_{p,q,x}^{(k)}\mathcal{E}_{n,p,q}(x)=\frac{p^{\binom{k}{2}}[n]_{p,q}!}{[n-k]_{p,q}!}\mathcal{E}_{n-k,p,q}(p^kx).$$

Theorem 2.3. *The (p,q)-Euler polynomial is a solution of the following (p,q)-differential equation:*

$$\begin{aligned}&\frac{1}{[n]_{p,q}!}D_{p,q,x}^{(n)}\mathcal{E}_{n,p,q}(p^nx)+\frac{1}{[n-1]_{p,q}!}D_{p,q,x}^{(n-1)}\mathcal{E}_{n,p,q}(p^{-(n-1)}x)+\cdots\\&+\frac{1}{[2]_{p,q}!}D_{p,q,x}^{(2)}\mathcal{E}_{n,p,q}(p^{-2}x)+D_{p,q,x}^{(1)}\mathcal{E}_{n,p,q}(p^{-1}x)+\mathcal{E}_{n,p,q}(p^{-1}x)\\&+\mathcal{E}_{n,p,q}(x)-2p^{\binom{n}{2}}x^n=0.\end{aligned}$$

Proof. We suppose that $e_{p,q} \neq -1$ in the generating function of the (p,q)-Euler polynomials. Then, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathcal{E}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} \left(\sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{t^n}{[n]_{p,q}!} + 1 \right) \\ &= 2 \sum_{n=0}^{\infty} p^{\binom{n}{2}} x^n \frac{t^n}{[n]_{p,q}!}. \end{aligned} \tag{2}$$

The left-hand side of the Equation (2) is transformed as

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathcal{E}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} \left(\sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{t^n}{[n]_{p,q}!} + 1 \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} \mathcal{E}_{n-k,p,q}(x) + \mathcal{E}_{n,p,q}(x) \right) \frac{t^n}{[n]_{p,q}!}. \end{aligned}$$

By using the above equation in Equation (2), we obtain

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} \mathcal{E}_{n-k,p,q}(x) = 2p^{\binom{n}{2}} x^n - \mathcal{E}_{n,p,q}(x). \tag{3}$$

From Corollary 2.2, we find a relation between

$$\mathcal{E}_{n-k,p,q}(x)$$

and

$$D_{p,q,x}^{(k)} \mathcal{E}_{n,p,q}(x)$$

as

$$\mathcal{E}_{n-k,p,q}(p^k x) = \frac{[n-k]_{p,q}!}{p^{\binom{k}{2}} [n]_{p,q}!} D_{p,q,x}^{(k)} \mathcal{E}_{n,p,q}(x). \tag{4}$$

Substituting the Equation (4) in (3), we have

$$\sum_{k=0}^{n} \frac{1}{[k]_{p,q}!} D_{p,q,x}^{(k)} \mathcal{E}_{n,p,q}(p^{-k} x) + \mathcal{E}_{n,p,q}(x) - 2p^{\binom{n}{2}} x^n = 0.$$

From the above equation, we obtain the required result. □

Corollary 2.4. *Set $p = 1$ in Theorem 2.3. Then, the following holds*

$$\begin{aligned}&\frac{1}{[n]_q!}D_{q,x}^{(n)}\mathcal{E}_{n,q}(x)+\frac{1}{[n-1]_q!}D_{q,x}^{(n-1)}\mathcal{E}_{n,q}(x)+\cdots\\&+\frac{1}{[2]_q!}D_{q,x}^{(2)}\mathcal{E}_{n,q}(x)+D_{q,x}^{(1)}\mathcal{E}_{n,q}(x)+2\mathcal{E}_{n,q}(x)-2x^n=0,\end{aligned}$$

where $[n]_q$ is the q-number, D_q is the q-derivative, and $\mathcal{E}_{n,q}(x)$ is the q-Euler polynomial.

Theorem 2.5. *For $\alpha \neq 0$, $\beta \neq 0$, we have the following:*

$$\begin{aligned}&\frac{\beta^n\mathcal{E}_{n,p,q}(\beta^{-1}y)}{[n]_{p,q}!p^{\binom{n}{2}}}D_{p,q,x}^{(n)}\mathcal{E}_{n,p,q}(\alpha^{-1}p^{-n}x)\\&+\frac{\beta^{n-1}\mathcal{E}_{n-1,p,q}(\beta^{-1}y)}{\alpha^{-1}[n-1]_{p,q}!p^{\binom{n-1}{2}}}D_{p,q,x}^{(n-1)}\mathcal{E}_{n,p,q}(\alpha^{-1}p^{-(n-1)}x)+\cdots\\&+\frac{\beta\mathcal{E}_{1,p,q}(\beta^{-1}y)}{\alpha^{1-n}}D_{p,q,x}^{(1)}\mathcal{E}_{n,p,q}(\alpha^{-1}p^{-1}x)+\frac{\mathcal{E}_{0,p,q}(\beta^{-1}y)}{\alpha^{-n}}\mathcal{E}_{n,p,q}(\alpha^{-1}x)\\&=\frac{\alpha^n\mathcal{E}_{n,p,q}(\alpha^{-1}y)}{[n]_{p,q}!p^{\binom{n}{2}}}D_{p,q,x}^{(n)}\mathcal{E}_{n,p,q}(\beta^{-1}p^{-n}x)\\&+\frac{\alpha^{n-1}\mathcal{E}_{n-1,p,q}(\alpha^{-1}y)}{\beta^{-1}[n-1]_{p,q}!p^{\binom{n-1}{2}}}D_{p,q,x}^{(n-1)}\mathcal{E}_{n,p,q}(\beta^{-1}p^{-(n-1)}x)+\cdots\\&+\frac{\alpha\mathcal{E}_{1,p,q}(\alpha^{-1}y)}{\beta^{1-n}}D_{p,q,x}^{(1)}\mathcal{E}_{n,p,q}(\beta^{-1}p^{-1}x)+\frac{\mathcal{E}_{0,p,q}(\alpha^{-1}y)}{\beta^{-n}}\mathcal{E}_{n,p,q}(\beta^{-1}x).\end{aligned}$$

Proof. To find a basic symmetric property of (p, q)-differential equation related to (p, q)-Euler polynomials, we consider a form A as

$$A := \frac{4e_{p,q}(tx)e_{p,q}(ty)}{\left(e_{p,q}(\alpha t)+1\right)\left(e_{p,q}(\beta t)+1\right)}.$$

By applying (p, q)-Euler polynomials in form A, we derive

$$\begin{aligned}&\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\alpha^{n-k}\beta^k\mathcal{E}_{k,p,q}(\beta^{-1}y)\mathcal{E}_{n-k,p,q}(\alpha^{-1}x)\\&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\alpha^{k}\beta^{n-k}\mathcal{E}_{k,p,q}(\alpha^{-1}y)\mathcal{E}_{n-k,p,q}(\beta^{-1}x).\end{aligned}$$

The above equation can be transformed as

$$\sum_{k=0}^{n} \frac{\beta^k \mathcal{E}_{k,p,q}(\beta^{-1}y)}{\alpha^{k-n}[k]_{p,q}!p^{\binom{k}{2}}} D_{p,q,x}^{(k)} \mathcal{E}_{n,p,q}(\alpha^{-1}p^{-k}x)$$
$$= \sum_{k=0}^{n} \frac{\alpha^k \mathcal{E}_{k,p,q}(\alpha^{-1}y)}{\beta^{k-n}[k]_{p,q}!p^{\binom{k}{2}}} D_{p,q,x}^{(k)} \mathcal{E}_{n,p,q}(\beta^{-1}p^{-k}x).$$

Therefore, we find the desired result. □

Corollary 2.6. *Set $p=1$ in Theorem 2.5. Then, the following holds*

$$\frac{\beta^n \mathcal{E}_{n,q}(\beta^{-1}y)}{[n]_q!} D_{q,x}^{(n)} \mathcal{E}_{n,q}(\alpha^{-1}x) + \frac{\beta^{n-1} \mathcal{E}_{n-1,q}(\beta^{-1}y)}{\alpha^{-1}[n-1]_q!} D_{q,x}^{(n-1)} \mathcal{E}_{n,q}(\alpha^{-1}x)$$
$$+ \cdots + \frac{\beta \mathcal{E}_{1,q}(\beta^{-1}y)}{\alpha^{1-n}} D_{q,x}^{(1)} \mathcal{E}_{n,q}(\alpha^{-1}x) + \frac{\mathcal{E}_{0,q}(\beta^{-1}y)}{\alpha^{-n}} \mathcal{E}_{n,q}(\alpha^{-1}x)$$
$$= \frac{\alpha^n \mathcal{E}_{n,q}(\alpha^{-1}y)}{[n]_q!} D_{q,x}^{(n)} \mathcal{E}_{n,q}(\beta^{-1}x)$$
$$+ \frac{\alpha^{n-1} \mathcal{E}_{n-1,q}(\alpha^{-1}y)}{\beta^{-1}[n-1]_q!} D_{q,x}^{(n-1)} \mathcal{E}_{n,q}(\beta^{-1}p^{-(n-1)}x)$$
$$+ \cdots + \frac{\alpha \mathcal{E}_{1,q}(\alpha^{-1}y)}{\beta^{1-n}} D_{q,x}^{(1)} \mathcal{E}_{n,q}(\beta^{-1}x) + \frac{\mathcal{E}_{0,q}(\alpha^{-1}y)}{\beta^{-n}} \mathcal{E}_{n,q}(\beta^{-1}x),$$

where $\mathcal{E}_{n,q}(x)$ is the q-Euler polynomial.

Corollary 2.7. *Putting $\alpha=1$ in Theorem 2.5, the following holds*

$$\frac{\beta^n \mathcal{E}_{n,p,q}(\beta^{-1}y)}{[n]_{p,q}!p^{\binom{n}{2}}} D_{p,q,x}^{(n)} \mathcal{E}_{n,p,q}(p^{-n}x)$$
$$+ \frac{\beta^{n-1} \mathcal{E}_{n-1,p,q}(\beta^{-1}y)}{[n-1]_{p,q}!p^{\binom{n-1}{2}}} D_{p,q,x}^{(n-1)} \mathcal{E}_{n,p,q}(p^{-(n-1)}x)$$
$$+ \cdots + \beta \mathcal{E}_{1,p,q}(\beta^{-1}y) D_{p,q,x}^{(1)} \mathcal{E}_{n,p,q}(p^{-1}x) + \mathcal{E}_{0,p,q}(\beta^{-1}y) \mathcal{E}_{n,p,q}(x)$$
$$= \frac{\mathcal{E}_{n,p,q}(y)}{[n]_{p,q}!p^{\binom{n}{2}}} D_{p,q,x}^{(n)} \mathcal{E}_{n,p,q}(\beta^{-1}p^{-n}x)$$
$$+ \frac{\mathcal{E}_{n-1,p,q}(y)}{\beta^{-1}[n-1]_{p,q}!p^{\binom{n-1}{2}}} D_{p,q,x}^{(n-1)} \mathcal{E}_{n,p,q}(\beta^{-1}p^{-(n-1)}x)$$
$$+ \cdots + \frac{\mathcal{E}_{1,p,q}(y)}{\beta^{1-n}} D_{p,q,x}^{(1)} \mathcal{E}_{n,p,q}(\beta^{-1}p^{-1}x) + \frac{\mathcal{E}_{0,p,q}(y)}{\beta^{-n}} \mathcal{E}_{n,p,q}(\beta^{-1}x).$$

Theorem 2.8. *A solution of the (p,q)-differential equation is given by*

$$\frac{p^n\mathcal{E}_{n,p,q}(1)}{p^{\binom{n}{2}}[n]_{p,q}!}D^{(n)}_{p,q,x}\mathcal{E}_{n,p,q}(p^{1-n}x)+\frac{p^{n-1}q\mathcal{E}_{n-1,p,q}(1)}{p^{\binom{n-1}{2}}[n-1]_{p,q}!}D^{(n-1)}_{p,q,x}\mathcal{E}_{n,p,q}(p^{2-n}x)$$
$$+\cdots+pq^{n-1}\mathcal{E}_{1,p,q}(1)D^{(1)}_{p,q,x}\mathcal{E}_{n,p,q}(x)+(\mathcal{E}_{0,p,q}(1)-2q)q^n\mathcal{E}_{n,p,q}(px)$$
$$+2\mathcal{E}_{n+1,p,q}(qx)=0$$

is the (p,q)-Euler polynomials.

Proof. By using (p,q)-derivative of a product and (p,q)-derivative of a ratio in the generating function of (p,q)-Euler polynomials, we have

$$\begin{aligned}
&D_{p,q,t}\sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}(qx)\frac{t^n}{[n]_{p,q}!}\\
&=e_{p,q}(pqtx)D_{p,q,t}\left(\frac{2}{e_{p,q}(t)+1}\right)+\frac{2}{e_{p,q}(qt)+1}D_{p,q,t}e_{p,q}(qtx)\\
&=e_{p,q}(pqtx)\left(\frac{2qx}{e_{p,q}(qt)+1}-\frac{2e_{p,q}(pt)}{(e_{p,q}(pt)+1)(e_{p,q}(qt)+1)}\right).
\end{aligned}$$

and note

$$e_{p,q}(pqtx)=\frac{e_{p,q}(qt)+1}{2}\sum_{n=0}^{\infty}q^n\mathcal{E}_{n,p,q}(px)\frac{t^n}{[n]_{p,q}!}.$$

From the above equations, we find

$$\begin{aligned}
&D_{p,q,t}\sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}(qx)\frac{t^n}{[n]_{p,q}!}\\
&=\sum_{n=0}^{\infty}q^n\mathcal{E}_{n,p,q}(px)\frac{t^n}{[n]_{p,q}!}\\
&\left(qx-\frac{1}{e_{p,q}(pt)+1}e_{p,q}(pt)\right)\\
&=\sum_{n=0}^{\infty}q^{n+1}x\mathcal{E}_{n,p,q}(px)\frac{t^n}{[n]_{p,q}!}\\
&-2^{-1}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_{p,q}p^lq^{n-l}\mathcal{E}_{l,p,q}(1)\mathcal{E}_{n-l,p,q}(px)\right)\frac{t^n}{[n]_{p,q}!}.
\end{aligned}$$

Therefore, the above equation can be changed as

$$\begin{aligned} & D_{p,q,t}\mathcal{E}_{n,p,q}(qx) \\ & = q^{n+1}x\mathcal{E}_{n,p,q}(px) - 2^{-1}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_{p,q} p^l q^{n-l}\mathcal{E}_{l,p,q}(1)\mathcal{E}_{n-l,p,q}(px). \end{aligned} \tag{5}$$

From Definition 1.5, we also have

$$D_{p,q,t}\mathcal{E}_{n,p,q}(qx) = \mathcal{E}_{n-l,p,q}(qx). \tag{6}$$

Combining the Equations (5) and (6), we find the following relation

$$\begin{aligned} & 2^{-1}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_{p,q} p^l q^{n-l}\mathcal{E}_{l,p,q}(1)\mathcal{E}_{n-l,p,q}(px) \\ & = q^{n+1}x\mathcal{E}_{n,p,q}(px) - \mathcal{E}_{n-l,p,q}(qx). \end{aligned} \tag{7}$$

From Corollary 2.2, we note that

$$\mathcal{E}_{n-l,p,q}(px) = \frac{[n-l]_{p,q}!}{p^{\binom{l}{2}}[n]_{p,q}!}D_{p,q,x}^{(l)}\mathcal{E}_{n,p,q}(p^{1-l}x). \tag{8}$$

Substituting the Equation (8) in (7), we derive

$$\sum_{l=0}^{n}\frac{p^l\mathcal{E}_{n,p,q}(1)}{p^{\binom{l}{2}}[l]_{p,q}!}D_{p,q,x}^{(l)}\mathcal{E}_{n,p,q}(p^{1-l}x) = 2(q^{n+1}x\mathcal{E}_{n,p,q}(px) - \mathcal{E}_{n-l,p,q}(qx)).$$

which obtain the required result. □

Corollary 2.9. *Putting $p=1$ in Theorem 2.8, we have*

$$\begin{aligned} & \frac{\mathcal{E}_{n,q}(1)}{[n]_q!}D_{q,x}^{(n)}\mathcal{E}_{n,q}(x) + \frac{q\mathcal{E}_{n-1,q}(1)}{[n-1]_q!}D_{q,x}^{(n-1)}\mathcal{E}_{n,q}(x) + \cdots \\ & + q^{n-1}\mathcal{E}_{1,q}(1)D_{q,x}^{(1)}\mathcal{E}_{n,q}(x) + (\mathcal{E}_{0,q}(1) - 2q)q^n\mathcal{E}_{n,q}(x) + 2\mathcal{E}_{n+1,q}(qx) = 0. \end{aligned}$$

where $\mathcal{E}_{n,q}(x)$ is the q-Euler polynomial.

3. Conclusion

We obtained a k-order differential equation by using a relationship between (p,q)-Euler numbers and (p,q)-Euler polynomials. We derived several higher-order differential equations with the (p, q)-Euler polynomial as the solution from a relation between the (p, q)-Euler polynomials and the k-order differential equation. We also found the symmetric structure of higher order differential equations.

Acknowledgement. The authors would like to express their thanks to the anonymous referees for reading this paper and consequently their comments and suggestions.

References

[1] S. Araci, U. Duran, M. Acikgoz, H. M. Srivastava, *A certain (p, q)-derivative operator and associated divided differences*, J. Inequal and Appl., 301 (2016), 1-8.

[2] G. Brodimas, A. Jannussis, R. Mignani, *Two-parameter quantum groups*, Universita di Roma Preprint, Nr. 820 (1991), 16 pages.

[3] R. Chakrabarti, R. Jagannathan, *A (p, q)-oscillator realization of two-parameter quantum algebras*, J. Phys. A: Math. Gen., 24 (1991), L711.

[4] J. L. Cieslinski, *Improved q-exponential and q-trigonometric functions*, arXiv:1006.5652v1 [math.CA], (2010), 2110-2114.

[5] R.B. Corcino, *On (P, Q)-Binomial coefficients*, Electron. J. Combin. Number Theory, 8 (2008), #A29, 1-16.

[6] U. Duran, M. Acikgoz, S. Araci, *On (p, q)-Bernoulli, (p, q)-Euler and (p, q)-Genocchi Polynomials*, J. Comp. and Theo. Nano., November 2016, 7833-7846.

[7] R. Jagannathan, K. S. Rao, *Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series*, Pro-

ceeding of the International Conference on Number Theory and Mathematical Physics, Srinivasa Ramanujan Centre, Kumbakonam, India, 20-21 December 2005.

[8] R. Jagannathan, (p, q)-*Special functions*, Special Functions and Differential Equations, Proceedings of a Workshop held at The Institute of Mathematical Sciences, Matras, India, January (1997), 13-24.

[9] C. S. Ryoo, J. Y. Kang, *Various Types of q-Differential Equations of Higher Order for q-Euler and q-Genocchi Polynomials*, Mathematics, 10 (2022), 1-16.

[10] C. S. Ryoo, *Some Properties of Degenerate Carlitz-type Twisted q-Euler Numbers and Polynomials*, Journal of Applied Mathematics and Informatics, 39 (2021), 1-11.

[11] N. Saba and A. Boussayoud, *New Theorem on Symmetric Functions and Their Applications on Some (p, q)-numbers*, Journal of Applied Mathematics and Informatics, 40 (2022), 243-258.

[12] P. N. Sadjang, *On the fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas*, arXiv:1309.3934 [math.QA] (2013).

[13] M. Wachs, D. White, (p, q)-*Stirling numbers and set partition statistics*, J. Conbin. Theory, A 56 (1991), 27-46.

Department of Mathematics Education
Hannam University
Daejeon, 34430
Republic of South Korea
E-mail: profyu@hnu.kr

Department of Mathematics Education
Silla University
Busan, 46958
Republic of South Korea
E-mail: jykang@silla.ac.kr

(*Received: December, 2022; Revised: January, 2023*)